THE MODULI SPACE OF GENERALIZED MORSE FUNCTIONS

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Abstract. We study the moduli and determine a homotopy type of the space of all generalized Morse functions on d-manifolds for given d. This moduli space is closely connected to the moduli space of all Morse functions studied in [11] and the classifying space of the corresponding cobordism category.

1. Introduction

Given a smooth compact manifold M^d and a fixed smooth function $\varphi : M^d \to \mathbf{R}$, let $\mathcal{G}(M^d, \varphi)$ denote the space of generalized Morse functions $f : M \to \mathbf{R}$ which agrees with φ in a neighbourhood of the boundary ∂M. This space (in the Whitney topology) satisfies an h-principle in the sense of Gromov, [5]. Here precisely we define $h\mathcal{G}(M^d, \varphi)$ to be the space of sections of the bundle $J^3_{\mathrm{gmf}}(M)$ of generalized Morse 3-jets that agrees with $j^3\varphi$ near ∂M. Taking the 3-jet of a generalized Morse function defines a map

$$j^3 : \mathcal{G}(M^d, \varphi) \longrightarrow h\mathcal{G}(M^d, \varphi) \ . \tag{1}$$

This map was first considered by Igusa in [6]. He proved that the map j^3 in (1) is d-connected and in [7] he calculated the "d-homotopy type" of $h\mathcal{G}(M^d, \varphi)$ by exhibiting a d-connected map

$$h\mathcal{G}(M^d, \varphi) \longrightarrow \Omega^\infty S^\infty(BO_+ \wedge M^d), \tag{2}$$

thus determining the d-homotopy type of the space $\mathcal{G}(M^d, \varphi)$.

Eliashberg and Mishachev [2, 3] and Vassiliev [13] showed that the map in (1) is actually a homotopy equivalence rather than just being d-connected. This is the starting point for this paper.

We study the moduli space of all generalized Morse functions on d-manifolds, i.e. the space of $\mathcal{G}(M^d, \varphi)$ as (M^d, φ) varies. There can be several candidates for such a moduli space. The one we present below is closely connected to the "moduli space" of all Morse functions considered in Section 4 of [11]. Indeed, the present note can be viewed as an addition to [11].

In Section 2 below we give the precise definition of our moduli space, and in Section 3 we determine its homotopy type, following the argument from [11].

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2. Definitions and results

2.1. **The moduli space.** Let $J^3(\mathbf{R}^d)$ be the space of 3-jets of smooth functions on $\mathbf{R}^d$,

$$p(x) = c + \ell(x) + q(x) + r(x) \ ,$$

where c is a constant, $\ell(x)$ is linear, $q(x)$ quadratic and $r(x)$ cubic,

$$\ell(x) = \sum_i a_i x_i \ , \quad q(x) = \sum_{ij} a_{ij} x_i x_j \ , \quad r(x) = \sum_{ijk} a_{ijk} x_i x_j x_k \ ,$$

with the coefficients a_{ij}, a_{ijk} symmetric in the indices.

Let $J^3_{\mathrm{gmf}}(\mathbf{R}^d) \subset J^3(\mathbf{R}^d)$ be the subspace of $p \in J^3(\mathbf{R}^d)$ such that one the following holds:

(i) $0 \in \mathbf{R}^d$ is not a critical point of p ($\ell \neq 0$);
(ii) $0 \in \mathbf{R}^d$ is a non-degenerate critical point of p ($\ell = 0$ and q is non-degenerate);
(iii) $0 \in \mathbf{R}^d$ is a birth-death singularity of p ($q : \mathbf{R}^d \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^d, \mathbf{R})$ has 1-dimensional kernal on which $r(x)$ is non-trivial).

The space $J^3_{\mathrm{gmf}}(\mathbf{R}^d)$ is invariant under the $O(d)$-action on the space $J^3(\mathbf{R}^d)$. Given a smooth manifold M^d with a metric, let $\mathcal{P}(M^d) \to M$ be the principal $O(d)$-bundle of orthogonal frames in the tangent bundle TM^d. Then

$$J^3_{\mathrm{gmf}}(TM^d) = \mathcal{P}(M^d) \times_{O(d)} J^3_{\mathrm{gmf}}(\mathbf{R}^d)$$

is a smooth fiber bundle on M, a subbundle of

$$J^3(TM^d) = \mathcal{P}(M^d) \times_{O(d)} J^3(\mathbf{R}^d).$$

Remark 2.1. Up to change of coordinates, a birth-death singularity is of the form

$$p(x) = x_1^3 - \sum_{j=2}^{i+1} x_j^2 + \sum_{k=i+2}^{d} x_k^2 \ .$$

The integer i is the Morse index of the quadratic form $q(x)$. In general, a quadratic form $q : \mathbf{R}^d \to \mathbf{R}$ induces a canonical decomposition

$$\mathbf{R}^d = V_-(q) \oplus V_0(q) \oplus V_+(q)$$

into negative eigenspace, the zero eigenspace and the positive eigenspace. In the case of generalized Morse jets, $\dim V_0(q)$ is either 0 or 1, and in the latter case the cubic term $r(x)$ restricts non-trivially to $V_0(q)$. The dimension of $V_-(q)$ is the index of the gmf-jet. $\diamond$

For a smooth manifold M^d, we have the 3-jet bundle $J^3(M, \mathbf{R}) \to M$ whose fiber $J^3(M, \mathbf{R})_x$ is the germ of 3-jets $(M, x) \to \mathbf{R}$, and the associated subbundle $J^3_{\mathrm{gmf}}(M, \mathbf{R}) \to M$. A choice of exponential function induces a fiber bundle isomorphism

$$J^3_{\mathrm{gmf}}(M, \mathbf{R}) \cong J^3_{\mathrm{gmf}}(TM), \tag{3}$$

and $f : M \to \mathbf{R}$ is a generalized Morse function precisely if $j^3(f) \in \Gamma(J^3_{\mathrm{gmf}}(M, \mathbf{R}))$, where we use $\Gamma(E)$ to denote the space of smooth sections of a vector bundle E.

Definition 2.1. Let X be a k-dimensional manifold without boundary. Let $\mathcal{J}_d(X)$ be the set of 4-tuples (E, π, f, j) of a $(k+d)$-manifold E with maps

$$(\pi, f, j) : E \longrightarrow X \times \mathbf{R} \times \mathbf{R}^{d-1+\infty}$$

subject to the conditions

(i) $(\pi, f) : E \longrightarrow X \times \mathbf{R}$ is a proper map;
(ii) $(f, j) : E \longrightarrow \mathbf{R} \times \mathbf{R}^{d-1+\infty}$ is an embedding;
(iii) $\pi : E \longrightarrow X$ is a submersion;
(iv) for any $x \in X$, the restriction $f_x = f|_{E_x} : E_x \longrightarrow \mathbf{R}$ to each fiber $E_x = \pi^{-1}(x)$ is a generalized Morse function.

In (ii) above $\mathbf{R}^{d-1+\infty}$ is the union or colimit of $\mathbf{R}^{d-1+N}$ as $N \to \infty$ and (ii) means that (f, j) embeds E into $\mathbf{R} \times \mathbf{R}^{d-1+N}$ for sufficiently large N. The definition above is the obvious analogue of Definition 2.7 of [11].

A smooth map $\phi : Y \longrightarrow X$ induces a pull-back

$$\phi^* : \mathcal{J}_d(X) \longrightarrow \mathcal{J}_d(Y), \quad \phi^* : (E, \pi, f, j) \mapsto (\phi^* E, \phi^* \pi, \phi^* f, \phi^* j) \quad \text{where}$$

$$\phi^* E = \{ \ (y, z) \in Y \times \mathbf{R}^{d+\infty} \mid (\phi(y), z) \in E \subset X \times \mathbf{R}^{d+\infty} \ \}$$

and the maps $\phi^*\pi$, $\phi^* f$ and $\phi^* j$ are given by corresponding projections from

$$\phi^* E \subset Y \times \mathbf{R} \times \mathbf{R}^{d-1+\infty}$$

on the factors Y, $\mathbf{R}$ and $\mathbf{R}^{d-1+\infty}$, respectively. In particular, we have $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ (rather than just being naturally equivalent), so the correspondence $X \mapsto \mathcal{J}_d(X)$ is a set-valued sheaf $\mathcal{J}_d$ on the category $\mathfrak{X}$ of smooth manifolds and smooth maps.

A set-valued sheaf on $\mathfrak{X}$ gives rise to a simplicial set $N_\bullet \mathcal{J}_d$ with

$$N_k \mathcal{J}_d = \mathcal{J}_d(\Delta^k_e), \quad \Delta^k_e = \{ \ (x_0, \dots, x_k) \in \mathbf{R}^{k+1} \mid \textstyle\sum x_i = 1 \ \}.$$

The geometric realization of $N_\bullet \mathcal{J}_d$ will be denoted by $|\mathcal{J}_d|$, and we make the following definition.

Definition 2.2. The moduli space of generalized Morse functions of d variables is the loop space $\Omega|\mathcal{J}_d|$.

Remark 2.2. If in Definition 2.1 we drop the assumption that $f : E \longrightarrow \mathbf{R}$ is a generalized Morse function, then $\mathcal{J}_d$ reduces to the sheaf $D_d = D_d(-, \infty)$ of [4, Definition 3.3] associated to the space of embedded d-manifolds, and $\Omega|D_d| \cong \Omega^\infty MT(d)$ by Theorem 3.4 of [4]. $\diamond$

Associated with the set valued sheaf $\mathcal{J}_d(X)$, $X \in \mathfrak{X}$, we have a sheaf $\mathcal{J}_d^{\mathcal{A}}(X)$ of partially ordered sets, i.e. a category valued sheaf, cf. [11, Section 4.2]. For connected X, an object of $\mathcal{J}_d^{\mathcal{A}}(X)$ consists of an element $(E, \pi, f, j) \in \mathcal{J}_d(X)$ together with an interval $A = [a_0, a_1] \subset \mathbf{R}$ subject to the condition that $f : E \longrightarrow \mathbf{R}$ be fiberwise transverse to ∂A (i.e. $\{a_0, a_1\}$ are regular values for each $f_x : E_x \longrightarrow \mathbf{R}$, $x \in X$). The partial ordering is given by

$$(E, \pi, f, j; A) \leq (E', \pi', f', j'; A') \text{ if}$$

$$(E, \pi, f, j) = (E', \pi', f', j') \text{ and } A \subset A' .$$

If X is not connected, $\mathcal{J}_d^{\mathcal{A}}(X)$ is the product of $\mathcal{J}_d^{\mathcal{A}}(X_j)$ over the connected components X_j.

We notice that each element $(E, \pi, j, f; A)$ restricts to a family of generalized Morse functions on a compact manifolds

$$(\pi, f, j) : f^{-1}(A) \hookrightarrow X \times [a_0, a_1] \times \mathbf{R}^{d-1+\infty},$$

where $A = [a_0, a_1]$. On the other hand, given such a family

$$(\pi, f, j) : E(A) \hookrightarrow X \times [a_0, a_1] \times \mathbf{R}^{d-1+\infty},$$

we can extend it to an element $(\hat{E}(A), \hat{\pi}, \hat{f}, \hat{j})$ by adding long collars:

$$\hat{E}(A) = (-\infty, a_0] \times f^{-1}(a_0) \cup E(A) \cup [a_1, \infty) \times f^{-1}(a_1).$$

If $(E(A)\pi, f, j)$ is a restriction of $(E, \pi, f, j) \in \mathcal{J}_d(X)$, then $(\hat{E}(A), \hat{\pi}, \hat{f}, \hat{j}; A)$ is concordant to (E, π, f, j) by [11, Lemma 2.19].

The forgetful map $\mathcal{J}_d^{\mathcal{A}}(X) \longrightarrow \mathcal{J}_d(X)$ is a map of category valued sheaves when we give $\mathcal{J}_d(X)$ the trivial category structure (with only identity morphisms). It induces a map

$$|\mathcal{J}_d^{\mathcal{A}}| \longrightarrow |\mathcal{J}_d|$$

of topological categories, and hence a map of their classifying spaces:

$$B|\mathcal{J}_d^{\mathcal{A}}| \longrightarrow B|\mathcal{J}_d| = |\mathcal{J}_d|,$$

where $B|\mathcal{J}| = |N_\bullet \mathcal{J}|$.

Theorem 2.1. *The map $B|\mathcal{J}_d^{\mathcal{A}}| \longrightarrow |\mathcal{J}_d|$ is a weak homotopy equivalence.*

Proof. This follows from [11, Theorem 4.2], which identifies $|B\mathcal{J}_d^{\mathcal{A}}|$ with $|\beta\mathcal{J}_d^{\mathcal{A}}|$, where $\beta\mathcal{J}_d^{\mathcal{A}}$ is a set-valued sheaf of [11, Definition 4.1], together with the analogue of [11, Proposition 4.10]: the map

$$|\beta\mathcal{J}_d^{\mathcal{A}}| \longrightarrow |\mathcal{J}_d^{\mathcal{A}}|$$

is a weak homotopy equivalence. Indeed, the proof of Proposition 4.10, which treats the case where $f : E \longrightarrow \mathbf{R}$ is a fiberwise Morse function (in a neighbourhood of $f^{-1}(0)$) carries over word by word to the situation of generalized Morse functions. $\square$

2.2. **The h-principle.** For a submersion $\pi : E \longrightarrow X$, $T^\pi E$ denotes the tangent bundle along the fibers. We can form the bundle

$$J^3_{\mathrm{gmf}}(T^\pi E) \longrightarrow E$$

of gmf-jets. Sections of this bundle will be denoted by $\hat{f}$, $\hat{g}, \dots$ etc. For given $z \in E$, the restriction $\hat{f}(z) : T(E_{\pi(z)}) \longrightarrow \mathbf{R}$ is a gmf-jet; its constant term will be denoted $f(z)$.

Definition 2.3. For a smooth manifold X, let $h\mathcal{J}_d(X)$ consists of maps

$$(\pi, f, j) : E \longrightarrow X \times \mathbf{R} \times \mathbf{R}^{d-1+\infty}$$

satisfying (i), (ii), and (iii) of Definition 2.1 together with a jet $\hat{f} \in \Gamma(J^3_{\mathrm{gmf}}(T^\pi E))$ having constant term f. □

A metric on $T^\pi E$ and an associated exponential map induces an isomorphism

$$J^3_\pi(E, \mathbf{R}) \xrightarrow{\cong} J^3(T^\pi E) \tag{4}$$

that sends gmf-jets to gmf-jets. Here $J^3_\pi(E, \mathbf{R}) \longrightarrow E$ is the fiberwise 3-jet bundle. Differentiation in the fiber direction only, defines a map

$$j^3_\pi : C^\infty(E, \mathbf{R}) \longrightarrow J^3_\pi(E, \mathbf{R})$$

which sends fiberwise generalized Morse functions into gmf-jets. This induces a map of sheaves

$$j^3_\pi : \mathcal{J}_d(X) \longrightarrow h\mathcal{J}_d(X),$$

and hence a map of their nerves

$$j^3_\pi : |\mathcal{J}_d| \longrightarrow |h\mathcal{J}_d|.$$

Using the sheaves $\mathcal{J}^{\mathcal{A}}_d$ and the associated $h\mathcal{J}^{\mathcal{A}}_d$, together with the h-principle of [3, 13], the argument of [11, Proposition 4.17] proves that the induced map

$$|\beta\mathcal{J}^{\mathcal{A}}_d| \longrightarrow |\beta h\mathcal{J}^{\mathcal{A}}_d|$$

is a weak homotopy equivalence. Finally, since the forgetful maps

$$|\beta\mathcal{J}^{\mathcal{A}}_d| \longrightarrow |\mathcal{J}^{\mathcal{A}}_d|, \quad |\beta h\mathcal{J}^{\mathcal{A}}_d| \longrightarrow |h\mathcal{J}^{\mathcal{A}}_d|$$

are weak homotopy equivalences by [11, Proposition 4.10], we get

Theorem 2.2. *The map $j^3_\pi : |\mathcal{J}_d| \longrightarrow |h\mathcal{J}_d|$ is weak homotopy equivalence.* □

3. Homotopy type of the moduli space

3.1. **The space $|h\mathcal{J}_d|$.** For any set-valued sheaf $\mathcal{F} : \mathfrak{X} \longrightarrow \mathcal{S}\mathrm{ets}$, let $\mathcal{F}[X]$ denote the set of concordance classes: $s_0, s_1 \in \mathcal{F}(X)$ are *concordant* ($s_0 \sim s_1$) if there exists an element $s \in \mathcal{F}(X \times \mathbf{R})$ such that $pr_X^*(s_0)$, $pr_X : X \times \mathbf{R} \longrightarrow X$, agrees with s on an open neighborhood of $X \times (-\infty, 0]$ and $pr_X^*(s_1)$ agrees with s on an open neighborhood of $X \times [1, \infty)$. The relation to the space $|\mathcal{F}|$ is given by

$$[X, |\mathcal{F}|] \cong \mathcal{F}[X]. \tag{5}$$

Let $G(d,n)$ denote the Grassmannian of d-planes in $\mathbf{R}^{d+n}$, and $G^{\mathrm{gmf}}(d,n)$ the space of pairs (V, f) with $V \in G(d,n)$ and $f : V \to \mathbf{R}$ a generalized Morse function with $f(0) = 0$. The space

$$U_{d,n}^{\perp} = \{ \ (v, V) \in \mathbf{R}^{d+n} \times G(d,n) \mid v \perp V \ \}$$

is an n-dimensional vector bundle on $G(d,n)$. Let $V_{d,n}^{\perp}$ be its pull-back along the forgetful map $G^{\mathrm{gmf}}(d,n) \longrightarrow G(d,n)$:

$$\begin{tikzcd} V_{d,n}^{\perp} \arrow[r] \arrow[d] & U_{d,n}^{\perp} \arrow[d] \\ G^{\mathrm{gmf}}(d,n) \arrow[r] & G(d,n) \end{tikzcd}$$

Similarly, we have canonical d-dimensional vector bundles $U_{d,n}$ and $V_{d,n}$ on $G(d,n)$ and $G^{\mathrm{gmf}}(d,n)$ respectively. The Thom spaces of the bundles $U_{d,n}^{\perp}$ and $V_{d,n}^{\perp}$ give rise to spectra $MT(d)$ and $MT^{\mathrm{gmf}}(d)$ which in degrees $(d+n)$ are

$$MT(d)_{d+n} = \mathsf{Th}(U_{d,n}^{\perp}), \qquad MT^{\mathrm{gmf}}(d)_{d+n} = \mathsf{Th}(V_{d,n}^{\perp}). \tag{6}$$

The infinite loop space of the spectrum $MT^{\mathrm{gmf}}(d)$ is defined to be

$$\Omega^{\infty} MT^{\mathrm{gmf}}(d) = \operatorname*{colim}_{n} \Omega^{d+n} \mathsf{Th}(V_{d,n}^{\perp}) \tag{7}$$

Theorem 3.1. *There is a weak homotopy equivalence*

$$\Omega |h\mathcal{J}_d| \simeq \Omega^{\infty} MT^{\mathrm{gmf}}(d). \tag{8}$$

Proof. This is completely similar to the proof of [11, Theorem 3.5] for the case of Morse functions using only transversality and the submersion theorem, [12]. □

We have left to examine the right-hand side of the equivalence (8). The results are similar in spirit to ones in [11, Section 3.1].

Let $\Sigma^{\mathrm{gmf}}(d,n) \subset G^{\mathrm{gmf}}(d,n)$ be the singular set of pairs (V, f) with $f : V \longrightarrow \mathbf{R}$ having vanishing linear term, i.e. $Df(0) = 0$. Consider the nonsingular set $G^{\mathrm{gmf}}(d,n) \setminus \Sigma^{\mathrm{gmf}}(d,n)$ given as set of pairs $(V, f) \in G^{\mathrm{gmf}}(d,n)$ with $f = \ell + q + r$ and $\ell \neq 0$. We notice that the space $G^{\mathrm{gmf}}(d,n) \setminus \Sigma^{\mathrm{gmf}}(d,n)$ retracts to the space

$$\hat{G}^{\mathrm{gmf}}(d,n) = \{ \ (V, f) \in G^{\mathrm{gmf}}(d,n) \mid f = \ell + q + r, \ \text{with } |\ell| = 1, \ q = 0, \ r = 0 \},$$

where $|\ell|$ is a norm of the linear part $\ell : V \longrightarrow \mathbf{R}$.

Lemma 3.2. *There is a homeomorphism*

$$\hat{G}^{\mathrm{gmf}}(d,n) \cong O(d+n)/\left(O(d-1)\times O(n)\right) .$$

Proof. For a pair $(V,f) \in \hat{G}^{\mathrm{gmf}}(d,n)$ we have $V \in G(d,n)$ and $f = \ell$ with $|\ell| = 1$. We may think of ℓ as a linear projection on the first coordinate, which is the same as to say that the space V contains a subspace

$$\mathbf{R}\times\{0\}\times\{0\} \subset \mathbf{R}\times\mathbf{R}^{d-1}\times\mathbf{R}^{n} \tag{9}$$

with ℓ being a projection on it. This identifies $\hat{G}^{\mathrm{gmf}}(d,n)$ with the homogeneous space $O(d+n)/\left(O(d-1)\times O(n)\right)$. ☐

Since $G(d-1,n) = O(d-1+n)/(O(d-1)\times O(n))$, we observe that the map

$$i_{d,n} : G(d-1,n) \longrightarrow \hat{G}^{\mathrm{gmf}}(d,n)$$

is $(d+n-2)$-connected and that $i^*_{d,n}(V^{\perp}_{d,n}|_{\hat{G}^{\mathrm{gmf}}(d,n)}) \cong U^{\perp}_{d-1,n}$.

On the other hand, $\Sigma^{\mathrm{gmf}}(d,n) \subset G^{\mathrm{gmf}}(d,n)$ has normal bundle $V^*_{d,n} \cong V_{d,n}$ and the inclusion

$$(D(V_{d,n}), S(V_{d,n})) \longrightarrow (G^{\mathrm{gmf}}(d,n), G^{\mathrm{gmf}}(d,n)\setminus\Sigma^{\mathrm{gmf}}(d,n))$$

is an excision map. This leads to the cofibration

$$\mathsf{Th}(j^*V^{\perp}_{d,n}) \longrightarrow \mathsf{Th}(V^{\perp}_{d,n}) \longrightarrow \mathsf{Th}((V^{\perp}_{d,n}\oplus V_{d,n})|_{\Sigma^{\mathrm{gmf}}(d,n)}), \tag{10}$$

where j is the inclusion

$$j : G^{\mathrm{gmf}}(d,n)\setminus\Sigma^{\mathrm{gmf}}(d,n) \longrightarrow G^{\mathrm{gmf}}(d,n) .$$

By the above, there is $(2n+d-2)$-connected map

$$i_{d,n} : \mathsf{Th}(U^{\perp}_{d-1,n}) \longrightarrow \mathsf{Th}(V^{\perp}_{d,n}).$$

With the notation of (6), we get from (10) a cofibration of spectra

$$\Sigma^{-1}MT(d-1) \longrightarrow MT^{\mathrm{gmf}}(d) \longrightarrow \Sigma^{\infty}(\Sigma^{\mathrm{gmf}}(d,\infty)_+) \tag{11}$$

and the corresponding homotopy fibration sequence of infinite loop spaces

$$\Omega^{\infty}\Sigma^{-1}MT(d-1) \longrightarrow \Omega^{\infty}MT^{\mathrm{gmf}}(d) \longrightarrow \Omega^{\infty}\Sigma^{\infty}(\Sigma^{\mathrm{gmf}}(d,\infty)_+)$$

Remark 3.1. The main theorem of [4] asserts a homotopy equivalence

$$\Omega^{\infty}MT(d) \simeq \Omega B\mathcal{C}ob_d,$$

where $\mathcal{C}ob_d$ is the cobordism category of embedded manifolds: the objects are (M^{d-1},a) of a closed $(d-1)$-submanifold of $\{a\}\times\mathbf{R}^{\infty+d-1}$ and the morphisms are embedded cobordisms $W^d \subset [a_0,a_1]\times\mathbf{R}^{\infty+d-1}$ transversal at $\{a_i\}\times\mathbf{R}^{\infty+d-1}$. In particular, we have weak homotopy equivalence

$$\Omega^{\infty}\Sigma^{-1}MT(d-1) \simeq \Omega^2 B\mathcal{C}ob_{d-1}. \tag{12}$$

3.2. **The singularity space.** In [7], Igusa analyzed the singularity space $\Sigma^{\mathrm{gmf}}(d) \subset J^3_{\mathrm{gmf}}(\mathbf{R}^d)$ by decomposing it with respect to the Morse index. The result, stated in [7, Proposition 3.4], is as follows. Consider the homogeneous spaces

$$\begin{aligned} X^1(i) &= O(d) \Big/ O(i) \times O(1) \times O(d-i-1), \\ X(i) &= O(d) \Big/ O(i) \times O(d-i). \end{aligned}$$

and note that there are quotient maps

$$f_i : X^1(i) \to X(i), \quad g_i : X^1(i) \to X(i+1),$$

upon embedding $O(i)\times O(1)\times O(d-i-1)$ in $O(i)\times O(d-i)$ and in $O(i+1)\times O(d-i-1)$, respectively. These maps fit into the diagram

$$\mathcal{D}(d) = \left(\begin{tikzcd}[column sep=small] & X^1(0) \arrow[dl, "f_0"'] \arrow[dr, "g_0"] & & X^1(1) \arrow[dl, "f_1"'] \arrow[dr, "g_1"] & & \cdots \arrow[dl, "f_2"'] \arrow[dr, "g_{d-2}"] & & X^1(d-1) \arrow[dl, "f_{d-1}"'] \arrow[dr, "g_{d-1}"] & \\ X(0) & & X(1) & & X(2) & & X(d-1) & & X(d) \end{tikzcd} \right)$$

and [7, Proposition 3.4] states that the homotopy colimit of the diagram $\mathcal{D}(d)$ is homotopy equivalent to $\Sigma^{\mathrm{gmf}}(d)$

$$\Sigma^{\mathrm{gmf}}(d) \simeq \operatorname{hocolim} \mathcal{D}(d). \tag{13}$$

It is easy to see that there are homeomorphisms

$$\begin{aligned} G^{\mathrm{gmf}}(d,n) &= \Big(O(d+n) \Big/ O(n) \Big) \times_{O(d)} J^3_{\mathrm{gmf}}(\mathbf{R}^d), \\ \Sigma^{\mathrm{gmf}}(d,n) &= \Big(O(d+n) \Big/ O(n) \Big) \times_{O(d)} \Sigma^{\mathrm{gmf}}(d), \end{aligned}$$

and (13) implies that $\Sigma^{\mathrm{gmf}}(d,n)$ is homotopy equivalent to the homotopy colimit of the diagram

$$\mathcal{D}(d,n) = \Big(O(d+n) \Big/ O(n) \Big) \times_{O(d)} \mathcal{D}(d) \ .$$

For $n \to \infty$, the Stiefel manifold $O(n+d)/O(n)$ becomes contractible, and $\mathcal{D}(d,\infty)$ is the diagram

$$\begin{tikzcd}[column sep=small] & Y^1(0) \arrow[dl, "\bar f_0"'] \arrow[dr, "\bar g_0"] & & Y^1(1) \arrow[dl, "f_1"'] \arrow[dr, "\bar g_1"] & & \cdots \arrow[dl, "\bar f_2"'] \arrow[dr, "\bar g_{d-2}"] & & Y^1(d-1) \arrow[dl, "\bar f_{d-1}"'] \arrow[dr, "\bar g_{d-1}"] & \\ Y(0) & & Y(1) & & Y(2) & & Y(d-1) & & Y(d) \end{tikzcd} \tag{14}$$

with

$$\begin{aligned} Y^1(i) &= BO(i)\times BO(1)\times BO(d-i-1), \\ Y(i) &= BO(i)\times BO(d-i), \end{aligned}$$

and $\bar{f}_i$ and $\bar{g}_i$ the obvious maps. So $\Sigma^{\mathrm{gmf}}(d,\infty)$ is the homotopy colimit of (14).

We want to compare this to the singular set $\Sigma^{\mathrm{mf}}(d,\infty)$ which appears when one considers the moduli space of Morse functions rather than generalized Morse functions was calculated in [11, Lemma 3.1]:

$$\Sigma^{\mathrm{mf}}(d,n) \cong \prod_{i=0}^{d} \left[\Big(O(d+n)\Big/O(n)\Big)\times_{O(d)}\Big(O(d)\Big/O(i)\times O(d-i)\Big)\right]$$

so that

$$\Sigma^{\mathrm{mf}}(d,\infty)\cong\prod_{i=0}^{d} BO(i)\times BO(d-i)=\prod_{i=0}^{d} Y(i).$$

The cofiber of the map $\Sigma^{\mathrm{mf}}(d,\infty)\longrightarrow\Sigma^{\mathrm{gmf}}(d,\infty)$ is by (14) equal to the homotopy colimit of the diagram:

$$\begin{tikzcd}[column sep=small] & Y^1(0) \arrow[dl] \arrow[dr] & & Y^1(1) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & Y^1(d-1) \arrow[dl] \arrow[dr] & \\ * & & * & & * & & * & & * \end{tikzcd}$$

But this homotopy colimit is easy calculated to be

$$\bigvee_{i=0}^{d-1} S^1\wedge Y^1(i)_+\simeq\bigvee_{i=0}^{d-1} S^1\wedge(BO(i)\times BO(d-i-1))_+\ .$$

We get a cofibration of suspension spectra:

$$\Sigma^\infty(\Sigma^{\mathrm{mf}}(d,\infty)_+)\longrightarrow\Sigma^\infty(\Sigma^{\mathrm{gmf}}(d,\infty)_+)\longrightarrow\bigvee_{i=0}^{d-1}\Sigma^\infty(S^1\wedge(BO(i)\times BO(d-i-1)_+)).$$

Taking the associated infinite loop spaces we get

Proposition 3.3. *There is a homotopy fibration:*

$$\prod_{i=0}^{d-1}\Omega^\infty\Sigma^\infty(BO(i)\times BO(d-i-1)_+)\to\prod_{i=0}^{d}\Omega^\infty\Sigma^\infty(\Sigma^{\mathrm{mf}}(d,\infty)_+)\to\Omega^\infty\Sigma^\infty(\Sigma^{\mathrm{gmf}}(d,\infty)_+).\ \square$$

The constant map $\mathcal{D}(d)\to *$ into the constant diagram induces the map

$$\mathcal{D}(d,n)\to\Big(O(d+n)\Big/O(n)\Big)\times_{O(d)} *\ ,$$

where the target space is homotopy equivalent to $G(d,n)$. For $n\to\infty$, this induces the fiber bundle

$$p:\Sigma^{\mathrm{gmf}}(d,\infty)\to BO(d) \tag{15}$$

with the fiber $\Sigma^{\mathrm{gmf}}(d)$. We obtain the commutative diagram of cofibrations:

$$\begin{tikzcd}
\Sigma^{-1}MT(d-1) \arrow[r] \arrow[d, "Id"'] & MT^{\mathrm{gmf}}(d) \arrow[r] \arrow[d, "F"'] & \Sigma^{\infty}(\Sigma^{\mathrm{gmf}}(d,\infty)_+) \arrow[d, "\Sigma^{\infty}p"'] \\
\Sigma^{-1}MT(d-1) \arrow[r] & MT(d) \arrow[r] & \Sigma^{\infty}(BO(d)_+)
\end{tikzcd}$$

and a corresponding diagram of homotopy fibrations:

$$\begin{tikzcd}
\Omega^{\infty}\Sigma^{-1}MT(d-1) \arrow[r] \arrow[d, "Id"'] & \Omega^{\infty}MT^{\mathrm{gmf}}(d) \arrow[r] \arrow[d, "\Omega^{\infty}F"'] & \Omega^{\infty}\Sigma^{\infty}(\Sigma^{\mathrm{gmf}}(d,\infty)_+) \arrow[d, "\Omega^{\infty}\Sigma^{\infty}p"'] \\
\Omega^{\infty}\Sigma^{-1}MT(d-1) \arrow[r] & \Omega^{\infty}MT(d) \arrow[r] & \Omega^{\infty}\Sigma^{\infty}(BO(d)_+)
\end{tikzcd} \tag{16}$$

Since $\Sigma^{\infty}(\Sigma^{\mathrm{gmf}}(d,\infty)_+)$ and $\Sigma^{\infty}(BO(d)_+)$ are (-1)-connected, we obtain that the forgetful map $F: MT^{\mathrm{gmf}}(d)\to MT(d)$ induces isomorphism

$$\pi_{-i}MT^{\mathrm{gmf}}(d)\cong \pi_{-i}MT(d), \quad i\geq 0.$$

We consider the forgetful map $\theta^{\mathrm{gmf}}: G^{\mathrm{gmf}}(d,\infty)\longrightarrow G(d,\infty)$ as a *structure on d-dimensional bundles*. Then we denote by $\mathcal{C}ob_d^{\mathrm{gmf}}$ the category $\mathcal{C}ob_d^{\theta^{\mathrm{gmf}}}$ (see (5.3) and (5.4) of [4]) of manifolds (objects) and cobordisms (morphisms) equipped with a tangential θ^{gmf}-structure. Then the main theorem of [4] gives the following result:

Corollary 3.4. *There is weak homotopy equivalence*

$$B\mathcal{C}ob_d^{\mathrm{gmf}}\cong \Omega^{\infty-1}MT^{\mathrm{gmf}}(d),$$

and the forgetting map $B\mathcal{C}ob_d^{\mathrm{gmf}}\to B\mathcal{C}ob_d$ induces isomorphism

$$\Omega_d^{\mathrm{gmf}}=\pi_0 B\mathcal{C}ob_d^{\mathrm{gmf}}\simeq \pi_0 B\mathcal{C}ob_d=\Omega_d,$$

where Ω_d^{gmf} and Ω_d are corresponding cobordism groups.

3.3. **Remarks on the moduli space of Morse functions.** The paper [11] studied the moduli space of fiberwise Morse functions. The fibers are the space of functions which locally has 2-jets of the form $f:\mathbf{R}^d\to\mathbf{R}$, $f=f(0)+\ell(x)+q(x)$ subject to the conditions:

(i) $f(0)\neq 0$ or
(ii) $f(0)=0$ and $\ell(x)\neq 0$ or
(iii) $f(0)=0$, $\ell(x)=0$ and $q(x)$ is non-singular quadratic form.

The associated sheaf $\mathcal{J}_d^{\mathrm{mf}}(X)$, denoted by $\mathcal{W}(X)$ in [11], consists of maps

$$(\pi, f, j) : E \longrightarrow X \times \mathbf{R} \times \mathbf{R}^{d-1+\infty} \quad \text{with}$$

(a) (π, f) is proper map,
(b) (f, j) is an embedding,
(c) $\pi : E \longrightarrow X$ is a submersion of relative dimension d,
(d) for $x \in X$, $f_x : E_x \to \mathbf{R}$ is "Morse", i.e. its 2-jet satisfies the conditions (i), (ii) and (iii).

The space $|\mathcal{J}_d^{\mathrm{mf}}| = |\mathcal{W}|$ was determined up to homotopy in Theorems 1.2 and 3.5 of [11]. We recall the results. Let $G^{\mathrm{mf}}(d, n)$ be the space of pairs

$$(V, f) \in G(d, n) \times J^2(V)$$

with f satisfying the above conditions (i), (ii) and (iii) and $f(0) = 0$. Let $\hat{U}_{d,n}^{\perp}$ be the canonical n-dimensional bundle over $G^{\mathrm{mf}}(d, n)$ and $MT^{\mathrm{mf}}(d)$ be the spectrum with

$$MT^{\mathrm{mf}}(d)_{d+n} = \mathsf{Th}(\hat{U}_{d,n}^{\perp}) \ .$$

Theorem 3.5. ([MW]) *There is a homotopy equivalence*

$$\Omega|\mathcal{J}_d^{\mathrm{mf}}| \cong \Omega^\infty MT^{\mathrm{mf}}(d) := \operatorname*{colim}_n \Omega^{d+n} \mathsf{Th}(\hat{U}_{d,n}^{\perp}) \ .$$

Analogous to (11), there is the cofibration of spectra

(17) $$\Sigma^{-1} MT(d-1) \longrightarrow MT^{\mathrm{mf}}(d) \longrightarrow \Sigma^\infty(\Sigma^{\mathrm{mf}}(d, \infty)_+)$$

The inclusion $\mathcal{J}_d^{\mathrm{mf}}(X) \longrightarrow \mathcal{J}_d^{\mathrm{gmf}}(X)$ induces a map

$$\Omega|\mathcal{J}_d^{\mathrm{mf}}| \longrightarrow \Omega|\mathcal{J}_d^{\mathrm{gmf}}|$$

of moduli spaces which can be examined upon comparing the (11) and (17) We have the homotopy commutative diagram of homotopy fibrations

(18) $$\begin{tikzcd} \Omega^\infty\Sigma^{-1}MT(d-1) \arrow[r] \arrow[d, "\cong"] & \Omega^\infty MT^{\mathrm{mf}}(d) \arrow[r] \arrow[d] & \Omega^\infty\Sigma^\infty(\Sigma^{\mathrm{mf}}(d,n)_+) \arrow[d] \\ \Omega^\infty\Sigma^{-1}MT(d-1) \arrow[r] & \Omega^\infty MT^{\mathrm{gmf}}(d) \arrow[r] & \Omega^\infty\Sigma^\infty(\Sigma^{\mathrm{gmf}}(d,n)_+) \end{tikzcd}$$

The middle vertical row can be identified with the map

$$\Omega|\mathcal{J}_d^{\mathrm{mf}}| \longrightarrow \Omega|\mathcal{J}_d^{\mathrm{gmf}}|$$

and the right-hand vertical row corresponds to the right-hand arrow of Proposition 3.3. This gives

Corollary 3.6. *There is a homotopy fibration*

$$\prod_{i=0}^{d-1} \Omega^\infty \Sigma^{-1}(BO(i)\times BO(d-i-1))_+ \longrightarrow \Omega|\mathcal{J}_d^{\mathrm{mf}}| \longrightarrow \Omega|\mathcal{J}_d^{\mathrm{gmf}}| \ . \qquad \square$$

3.4. **Generalization to tangential structures.** Let $\theta : B \to BO(d)$ be a Serre fibration thought as a *structure on d-dimensional vector bundles*: If $f : X \to BO(d)$ is a map classifying a vector bundle over X, then a map $\ell : X \to B$ such that $f = \theta\circ\ell$. For a given n, we define the space $G^{\theta,\mathrm{gmf}}(d,n)$ as the pull-back:

$$\begin{tikzcd} G^{\theta,\mathrm{gmf}}(d,n) \arrow[r] \arrow[d, "\theta_{d,n}"] & B \arrow[d, "\theta"] \\ G^{\mathrm{gmf}}(d,n) \arrow[r, "i_n"] & BO(d) \end{tikzcd} \tag{19}$$

where i_n is the composition of the forgetful map $G^{\mathrm{gmf}}(d,n)\to G(d,n)$ and the canonical embedding $i_n^o : G(d,n)\hookrightarrow G(d,\infty) = BO(d)$. To define a corresponding sheaf $\mathcal{J}_d^\theta$ of generalized Morse function on manifolds with tangential structure θ, we use Definition 2.1 but adding the requirement that the manifold E and corresponding fibers $E_x = \pi^{-1}(x)$ are equipped with the compatible tangential structures. Similarly the sheaf $h\mathcal{J}_d^\theta$ is well-defined and there is the corresponding map $j^3_\pi(\theta) : |\mathcal{J}_d^\theta| \longrightarrow |h\mathcal{J}_d^\theta|$. The following result is a generalition of Theorem 2.2 providing the h-principle:

Theorem 3.7. *The map*

$$j^3_\pi(\theta) : |\mathcal{J}_d^\theta| \longrightarrow |h\mathcal{J}_d^\theta|$$

is weak homotopy equivalence. $\square$

To describe the homotopy type of the moduli space $\Omega|\mathcal{J}_d^\theta|$, we consider the bundle diagram:

$$\begin{tikzcd} V_{d,n}^{\theta,\perp} \arrow[r] \arrow[d] & V_{d,n}^{\perp} \arrow[d] \\ G^{\theta,\mathrm{gmf}}(d,n) \arrow[r, "\theta_{d,n}"] & G^{\mathrm{gmf}}(d,n) \end{tikzcd}$$

where $V_{d,n}^{\theta,\perp}$ is a pull-back of $V_{d,n}^{\perp}$. Similarly, let $V_{d,n}^{\theta} \to G^{\theta,\mathrm{gmf}}(d,n)$ be the pull-back of the bundle $V_{d,n}\to G^{\mathrm{gmf}}(d,n)$. The Thom space of the bundle $V_{d,n}^{\theta,\perp}$ gives rise to the spectrum $MT^{\theta,\mathrm{gmf}}(d)$ which in degrees $(d+n)$ is

$$MT^{\theta,\mathrm{gmf}}(d)_{d+n} = \mathsf{Th}(V_{d,n}^{\theta,\perp}). \tag{20}$$

We have the following version of Theorem 3.1:

Theorem 3.8. *There is weak homotopy equivalence*

$$\Omega|h\mathcal{J}_d^\theta| \cong \Omega^\infty MT^{\theta,\mathrm{gmf}}(d). \qquad \square$$

We next examine the homotopy type of $\Omega^\infty MT^{\theta,\mathrm{gmf}}(d)$ in terms of the corresponding singular sets $\Sigma^{\theta,\mathrm{gmf}}(d,n)\subset G^{\theta,\mathrm{gmf}}(d,n)$. Define the θ-Grassmannian $G^\theta(d,n)$ as the pull-back:

$$\begin{tikzcd} G^\theta(d,n) \arrow[r] \arrow[d, "\theta_n"'] & B \arrow[d, "\theta"'] \\ G(d,n) \arrow[r, "i_n^o"] & BO(d) \end{tikzcd} \tag{21}$$

where $i_n^o : G(d,n)\hookrightarrow BO(n)$ is the canonical embedding,

$$G^\theta(d,n)=\{(V,b)\mid i_n^o(V)=\theta(b)\ \}\subset G(d,n)\times BO(d).$$

Then it is easy to identify $G^{\theta,\mathrm{gmf}}(d,n)$ with the subspace

$$G^{\theta,\mathrm{gmf}}(d,n)=\{\ (V,b,f)\mid (V,b)\in G^\theta(d,n),\ \ f\in J^3_{\mathrm{gmf}}(V)\ \}.$$

Let $\Sigma^{\theta,\mathrm{gmf}}(d,n)$ be the singular set of triples (V,b,f), where $f:V\to\mathbf{R}$ has vanishing linear term. The non-sigular subspace $G^{\theta,\mathrm{gmf}}(d,n)\setminus\Sigma^{\theta,\mathrm{gmf}}(d,n)$ is the set of triples (V,b,f) with $f=\ell+q+r$, where the linear part $\ell\neq 0$. By analogy with the space $\hat{G}^{\mathrm{gmf}}(d,n)$ from Lemma 3.2, we define

$$\hat{G}^{\theta,\mathrm{gmf}}(d,n)=\{\ (V,b,f)\mid f=\ell+q+r,\ |\ell|=1,\ q=0,\ r=0\ \}$$

and notice that the space $G^{\theta,\mathrm{gmf}}(d,n)\setminus\Sigma^{\theta,\mathrm{gmf}}(d,n)$ retracts to $\hat{G}^{\theta,\mathrm{gmf}}(d,n)$. By construction, the space $\hat{G}^{\theta,\mathrm{gmf}}(d,n)$ is the pull-back in the diagram:

$$\begin{tikzcd} \hat{G}^{\theta,\mathrm{gmf}}(d,n) \arrow[r] \arrow[d] & \hat{G}^{\mathrm{gmf}}(d,n) \arrow[d, "g_n"'] \\ G^\theta(d,n) \arrow[r, "\theta_n"] & G(d,n) \end{tikzcd}$$

where $\theta_n : G^\theta(d,n)\to G(d,n)$ is from (21) and g_n is a composition of the forgetting map and the inclusion:

$$g_n:\hat{G}^{\mathrm{gmf}}(d,n)\hookrightarrow G^{\mathrm{gmf}}(d,n)\to G(d,n)$$

Similarly to the above case, the normal bundle of the inclusion $\Sigma^{\theta,\mathrm{gmf}}(d,n)\hookrightarrow G^{\theta,\mathrm{gmf}}(d,n)$ coincides with $(V^\theta_{d,n})^*\cong V^\theta_{d,n}$ restricted to $\Sigma^{\theta,\mathrm{gmf}}(d,n)$, and again the inclusion

$$(D(V^\theta_{d,n}),S(V^\theta_{d,n}))\longrightarrow(G^{\theta,\mathrm{gmf}}(d,n),G^{\theta,\mathrm{gmf}}(d,n)\setminus\Sigma^{\theta,\mathrm{gmf}}(d,n))$$

is an excision map. Let $j_\theta : G^{\theta,\mathrm{gmf}}(d,n)\setminus\Sigma^{\theta,\mathrm{gmf}}(d,n)\to G^{\theta,\mathrm{gmf}}(d,n)$ be the inclusion. This leads to the cofibration

$$\mathsf{Th}(j_\theta^* V^{\theta,\perp}_{d,n})\to\mathsf{Th}(V^{\theta,\perp}_{d,n})\to\mathsf{Th}((V^{\theta,\perp}_{d,n}\oplus V^\theta_{d,n})_{\Sigma^{\theta,\mathrm{gmf}}(d,n)}) \tag{22}$$

and to the cofibration of spectra

$$\Sigma^{-1}MT^\theta(d-1)\to MT^{\theta,\mathrm{gmf}}(d)\to\Sigma^\infty(\Sigma^{\theta,\mathrm{gmf}}(d,\infty)_+)$$

with corresponding homotopy fibration of infinite loop spaces:

$$\Omega^\infty\Sigma^{-1}MT^\theta(d-1)\to\Omega^\infty MT^{\theta,\mathrm{gmf}}(d)\to\Omega^\infty\Sigma^\infty(\Sigma^{\theta,\mathrm{gmf}}(d,\infty)_+).$$

We denote by $\mathcal{C}ob^\theta$ the corresponding cobordism category of manifolds equipped with tangential structure θ and by $\mathcal{C}ob^{\theta,\mathrm{gmf}}$ the category with the condition that each morphism be equipped with a generalized Morse function as above.

Corollary 3.9. *There is weak homotopy equivalence*

$$B\mathcal{C}ob_d^{\theta,\mathrm{gmf}}\simeq\ \Omega^{\infty-1}MT^{\theta,\mathrm{gmf}}(d),$$

and the forgetful map $B\mathcal{C}ob_d^{\theta,\mathrm{gmf}}\to B\mathcal{C}ob_d^\theta$ *induces isomorphism*

$$\Omega_d^{\theta,\mathrm{gmf}}=\pi_0B\mathcal{C}ob_d^{\theta,\mathrm{gmf}}\cong\pi_0B\mathcal{C}ob_d=\Omega_d^\theta,$$

where $\Omega_d^{\theta,\mathrm{gmf}}$ *and* Ω_d^θ *are corresponding cobordism groups.*

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